Hidden symmetries of SD Poincaré Einstein metrics in split signature

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Holography aims to construct gravitational physics from $\mathscr{I}$.
Start from: [Math.DG/0504582, Duke Math (2007)], with C. LeBrun
$\rightsquigarrow$ *Global, Zoll(frei)* SD Poincaré-Einstein metrics.
Work with Giuseppe Bogna & Adam Kmec. [cf. also M. 2212.10895, Adamo, M. Sharma 2103.16984].
Responding to celestial symmetry algebras [Strominger '21, Tayler, Zhu '23].

Holography:

Slogan: Reformulate bulk physics in terms of ‘theory’ at conformal boundary at infinity.

Definition

A Poincaré-Einstein metric is an asymptotically hyperbolic metric (M^d, g) *with conformal compactification* $(\bar{M}, \bar{g})$

- $\bar{M} = M \cup \mathscr{I}, \qquad \mathscr{I} = \partial \bar{M}$
- $\bar{g} = \Omega^2 g$,
- $\mathscr{I} = \{\Omega = 0\}$, $d\Omega \neq 0$ *on* $\mathscr{I}$.
- *Ricci*$= -\Lambda g$

Problem: Use induced conformal structure on $\mathscr{I}$ as boundary data to reconstruct ‘Bulk’ (M, g).

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Global constructions (Euclidean):

- Poincaré-Einstein metrics on ball B^d from conformal structure on boundary $\mathscr{I} = S^{d-1}$, [Graham-Lee, 1992].

4d self-dual Poinaré-Einstein in split signature

Global models in split signature (conformal group $SO(3,3)/\mathbb{Z}_4$):

- Conformally flat models: $S^2 \times S^2$ or $S^2 \times S^2/\mathbb{Z}_2$:
$$ds^2 = \Omega^2(ds^2_{S^2_{\mathbf{x}}} - ds^2_{S^2_{\mathbf{y}}}) \,,$$
Coordinates $(\mathbf{x}, \mathbf{y}) \in \mathbb{R}^3 \times \mathbb{R}^3$, $|\mathbf{x}| = |\mathbf{y}| = 1$.
- $\mathbb{Z}_2$ acts by $(\mathbf{x}, \mathbf{y}) \to (-\mathbf{x}, -\mathbf{y})$.
- For $\Lambda \neq 0$: $\Omega = 1/y_3$, and $\mathscr{I} = S^2 \times S^1/\mathbb{Z}_2$.

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Curvature: for 4d manifold (M^4, g),

$$\Omega^2_M = \begin{pmatrix} \Omega^{2+} \\ \oplus \\ \Omega^{2-} \end{pmatrix}, \qquad \text{Riem} = \begin{pmatrix} \text{Weyl}^+ + S\delta & \text{Ricci}_0 \\ \text{Ricci}_0 & \text{Weyl}^- + S\delta \end{pmatrix}.$$

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This talk: focus on *self-dual* Poincaré-Einstein metrics:

$$\text{Ricci} = -\Lambda g\,, \qquad \text{Weyl}^- = 0\,, \qquad \text{(can allow } \Lambda \to 0 \text{ too).}$$

α and β-surfaces and the Zollfrei condition

The split signature conformally flat metric

$$ds^2 = \Omega^2(ds^2_{S^2_{\mathbf{x}}} - ds^2_{S^2_{\mathbf{y}}}),$$

admits a 3-parameter family of β-planes denoted by $\mathbb{PT}_{\mathbb{R}}$:

- respectively totally null ASD S^2s given by

$$\mathbf{x} = A\mathbf{y}, \qquad A \in SO(3) = \mathbb{RP}^3.$$

- Weyl$^- = 0 \Rightarrow \beta$-planes survive as β-surfaces.
- β-surfaces are projectively flat.
- If compact, β-surfaces are necessarily S^2 or $\mathbb{RP}^2$.
- Null geodesics are projectively $\mathbb{RP}^1$s or double cover.

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Following Guillemin we define:

Definition

An indefinite space (M^d, g) is (strongly) Zollfrei if all null geodesics are embedded S^1s (of same projective length).

Conformally self-dual case

Theorem (LeBrun & M. 2007)

Let (M^4, g) be Zollfrei with Weyl$^- = 0$. Then either

- $M = S^2 \times S^2/\mathbb{Z}_2 \Leftrightarrow$ *conformally flat, or*
- $M = S^2 \times S^2$ *and there is a* $1:1$*-correspondence between*
  1. *Perturbations of self-dual conformal structures* $[g]$*, and*
  2. *Deformations of the standard embedding of* $\mathbb{RP}^3 \subset \mathbb{CP}^3$ *modulo reparametrizations of* $\mathbb{RP}^3$ *and* $PGL(4, \mathbb{C})$ *on* $\mathbb{CP}^3$*.*

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Let $i\mathbb{R}^3 \times \mathbb{RP}^3 \subset \mathbb{CP}^3$ be a neighbourhood of of $\mathbb{RP}^3$ in $\mathbb{CP}^3$,

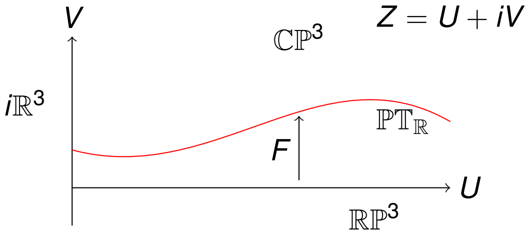


Figure: $\mathbb{PT}_{\mathbb{R}} = \{\text{graph } V = F(U)\}$ for some $F : \mathbb{RP}^3 \to \mathbb{R}^3$.

Key ideas of proof: 1. real geometry

- Zollfrei $\Rightarrow$ compact β-surfaces $\Rightarrow$ compactness of M.
- Two cases: β-surfaces are either S^2s or $\mathbb{RP}^2$s.
- Intersection properties: $\mathbb{RP}^2$s intersect in one point, S^2s in two $\rightsquigarrow$ characterisation of topology.
- The space of real β-surfaces $\mathbb{PT}_{\mathbb{R}}$ is constructed via

$$\mathscr{F}_{\mathbb{R}} = \{F \in \Omega^{2-}_M, F \wedge F = 0\}/\mathbb{R}^* \to M,$$

the S^1-bundle on M of ASD 2-plane elements with double fibration

$$\begin{array}{ccc} & \mathscr{F}_{\mathbb{R}} & \\ p \swarrow & & \searrow q \\ M^4 & & \mathbb{PT}_{\mathbb{R}}, \end{array}$$

- $\mathscr{F}_{\mathbb{R}}$ is foliated by β-surfaces and $\mathbb{PT}_{\mathbb{R}} \simeq \mathbb{RP}^3$ in both cases.

Key ideas: 2. Complex geometry

The fibrewise complexification of $\mathscr{F}_{\mathbb{R}}$

$$\mathscr{F}_{\mathbb{C}} = \{F \in \mathbb{C} \otimes \Omega^{2-}_M, F \wedge F = 0\}/\mathbb{C}^*,$$

is a $\mathbb{CP}^1$ bundle on M, fibre coord ζ.

- At each $x \in M$, $\mathscr{F}_{x\mathbb{R}}$ cuts $\mathscr{F}_{x\mathbb{C}}$ into two discs $D^{\pm}_x$.
- for $M = S^2 \times S^2$, $\exists$ global choice $\mathscr{F}^+_{\mathbb{C}}$, a D^+_x bundle on M.
- $\partial \mathscr{F}^+_{\mathbb{C}} = \mathscr{F}_{\mathbb{R}}$.
- $\mathscr{F}^+_{\mathbb{C}}$ admits a $\mathbb{C}$-involutive distribution $\mathcal{D} = \{\mathrm{Ker} F, \partial/\partial\bar{\zeta}\}$
  - defines a $\mathbb{C}$-structure on $\mathscr{F}^+_{\mathbb{C}} - \mathscr{F}_{\mathbb{R}}$ and
  - on $\mathscr{F}_{\mathbb{R}}$, leaves of $\mathcal{D} \cap \overline{\mathcal{D}} =$ lifts of β-surfaces.
- Blowing down $\mathscr{F}_{\mathbb{R}} \to \mathbb{PT}_{\mathbb{R}}$ yields compact complex manifold which must be $\mathbb{CP}^3 \supset \mathbb{PT}_{\mathbb{R}}$.
- If $M = S^2 \times S^2/\mathbb{Z}_2$, construct $\mathbb{CP}^3 \supset \mathbb{PT}_{\mathbb{R}}$ via double cover. Then $\mathbb{Z}_2 \rightsquigarrow$ complex conjugation fixing standard $\mathbb{RP}^3$.

3. Reconstruction of M from twistor space $\mathbb{PT}_{\mathbb{R}}$

Reconstruction:

Each $x \in M^4 \leftrightarrow$ holomorphic disc $\mathrm{D}_x^+ \subset \mathbb{CP}^3$ with $\partial \mathrm{D}_x \subset \mathbb{PT}_{\mathbb{R}}$:

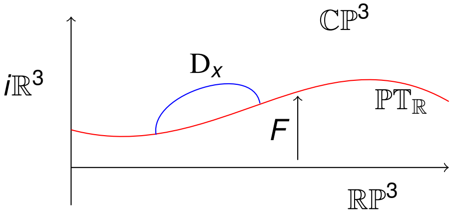


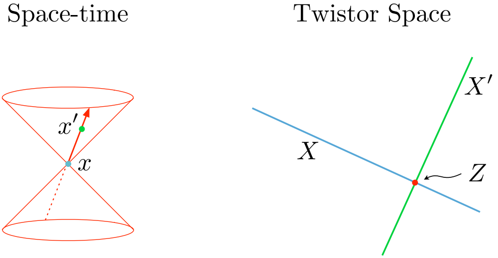
Figure: $\mathrm{D} =$ hol. disc $\subset \mathbb{CP}^3$ with $\partial \mathrm{D} \subset \mathbb{PT}_{\mathbb{R}}$.

- D_x^+ generates the degree-1 class in $H_2(\mathbb{CP}^3, \mathbb{PT}_{\mathbb{R}}, \mathbb{Z}) = \mathbb{Z}$.
- Reconstruct M from $\mathbb{PT}_{\mathbb{R}}$ space of all such disks:

$$M = \{\text{Moduli of degree-1 hol. disks: } D_x^+ \subset \mathbb{CP}^3, \partial D_x^+ \subset \mathbb{PT}_{\mathbb{R}}\}$$

- Finding such holomorphic discs is an elliptic problem of index 4, the problem is stable under deformations.
- Gives compact 4d moduli space M^4.

M^4 admits a conformal structure for which $\partial D^+_x \cap \partial D^+_{x'} = Z$ means that x, x' sit on same β-plane:



Restriction to Einstein case

Which $\mathbb{PT}_{\mathbb{R}} \subset \mathbb{CP}^3$ give SD Einstein $g \in [g]$ on $S^2 \times S^2$?

- Let Z^A, $A = 1, \ldots, 4$ be homogenous coordinates for $\mathbb{CP}^3$.
- Introduce real skew ε^{ABCD} and

$$I_{AB} = I_{[AB]}, \quad I^{AB} = \frac{1}{2}\varepsilon^{ABCD} I_{CD}, \quad \text{with} \quad I^{AB} I_{AC} = \Lambda \delta^B_C \,.$$

- To define contact and Poisson structures on $\mathbb{CP}^3$

$$\theta = I_{AB} Z^A dZ^B \in \Omega^1(2)\,, \qquad \{f, g\} := I^{AB} \frac{\partial f}{\partial Z^A} \frac{\partial g}{\partial Z^B}$$

valued in $\mathcal{O}(2)$, $\mathcal{O}(-2)$ respectively.

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  valued in $\mathcal{O}(2)$, $\mathcal{O}(-2)$ respectively.
- We have after Penrose & Ward:

Theorem

An Einstein $g \in [g]$ exists when $\theta|_{\mathbb{PT}_{\mathbb{R}}}$ and $\{,\}|_{\mathbb{PT}_{\mathbb{R}}}$ are real.

[A section of $\mathcal{O}(n)|_{\mathbb{PT}_{\mathbb{R}}}$ is real via $\mathcal{O}(-4)_{\mathbb{R}} = \Omega^3_{\mathbb{PT}_{\mathbb{R}}}$.]

Generating functions for Einstein embeddings

Explicitly in homogeneous coordinates:

- Let $Z^A = U^A + iV^A, U^A, V^A \in \mathbb{R}^4$.
- Let $H(U)$ be an arbtrary function of homogeneity degree 2,

$$U \cdot \frac{\partial H}{\partial U} = 2h.$$

- Then the Einstein conditions are satisfied by

$$\mathbb{T}_{\mathbb{R}} = \left\{ Z^A = U^A + iI^{AB} \frac{\partial H}{\partial U^B} \right\}$$

  projectivising gives $\mathbb{P}\mathbb{T}_{\mathbb{R}}$.
- All sufficiently small such arise in this way.

Split signature Poincaré-Einstein metrics

Are any corresponding Einstein metrics Poincaré-Einstein?

- They are small perturbations of the standard example

$$ds^2 = \frac{1}{y_3^2}(ds^2_{S^2_{\mathbf{x}}} - ds^2_{S^2_{\mathbf{y}}}) + \dots ,$$

  Coordinates $(\mathbf{x}, \mathbf{y}) \in \mathbb{R}^3 \times \mathbb{R}^3$, $|\mathbf{x}| = |\mathbf{y}| = 1$.
- Einstein scale encoded in solution to conformally invt

$$(\nabla_a \nabla_b + R_{ab})_0 \Phi = 0$$

  where Φ has conf. weight 1; scale to 1 for Einstein metric.
- Standard case $\Phi = y_3$, changes sign at $\mathscr{I}$.
- $\mathscr{I}$ cannot suddenly evaporate under perturbation.

Holography: SD Poincaré-Einstein Spaces from $\mathscr{I}$

Expectations:

- Data at $\mathscr{I}$ is its conformal structure:
  To fill in with SD Poincaré-Einstein, must be zollfrei.
- Two natural cases: $\mathscr{I} = S^2 \times S^1$ or $S^2 \times S^1/\mathbb{Z}_2$.
- Guillemin studied perturbations of $S^2 \times S^1/\mathbb{Z}_2$, his work $\Leftrightarrow$:

Proposition (Guillemin)

Zollfrei linear perturbations of $S^2 \times S^1/\mathbb{Z}_2$ conf. structure is

$$H^1_{\bar\partial}(\mathbb{CP}^3_0, \mathcal{O}(2)) = H^1_{\bar\partial}(\mathbb{CP}^3_+, \mathcal{O}(2)) \oplus H^1_{\bar\partial}(\mathbb{CP}^3_-, \mathcal{O}(2))$$

where $\mathbb{CP}^3_0 = \{I_{AB}Z^A\bar{Z}^B = 0\}$ *and* $\mathbb{CP}^3_\pm = \{\pm i I_{AB}Z^A\bar{Z}^B > 0\}$.

- Real such def'ms dont extend far off $\mathbb{CP}^3_0$ or $\mathscr{I}$, but

Proposition

for $\mathscr{I} = S^2 \times S^1$, $\exists$ extra class of perturbations from smooth sections h of $\mathcal{O}(2)$ over $\mathbb{RP}^3$ as above. These give complete fully nonlinear Zollfrei bulk metrics (on both sides).

- linear theory works by generalized 'X-ray transform'.

‘Hidden’ symmetries Following [Strominger 2021, Taylor, Zhu 2023]

Space of solutions as homogeneous space

Space $\mathcal{P}$ of Poincaré Einstein metrics on $S^2 \times S^2$ is mapped to

$$\mathcal{P} = \frac{\mathbb{C}\text{-Holomorphic Poisson diffeos near } \mathbb{PT}_{\mathbb{R}}}{\text{Real Poisson diffeos of } \mathbb{PT}_{\mathbb{R}}}$$
$$= \{\text{real } H \in C^\infty(\mathbb{RP}^3, \mathcal{O}(2))\} .$$

- $\mathcal{P}$ is homogeneous space.
- Infinitesimally Lie algebra $=$ complex $h \in C^\infty(\mathbb{RP}^3, \mathcal{O}(2))$
- Lie bracket $=$ Poisson bracket.
- acts by $\delta H = \Im h + \{\Re h, H\}$.
- To study $\Lambda \to 0$ let $Z^A = (\lambda_\alpha, \mu^{\dot\alpha})$, $\alpha = 0, 1$, $\dot\alpha = 0, 1$

$$I^{AB}\partial_{Z^A}\partial_{Z^B} = \varepsilon^{\dot\alpha\dot\beta}\partial_{\mu^{\dot\alpha}}\partial_{\mu^{\dot\beta}} + \Lambda\varepsilon^{\alpha\beta}\partial_{\lambda_\alpha}\partial_{\lambda_\beta} .$$

- As $\Lambda \to 0$, Lie algebra $= Lw_{1+\infty} =$ loop algebra of $Diff_0\mathbb{R}^2$.
- As found by Strominger via ‘celestial holography’ at $\Lambda = 0$,

From disks to scattering amplitudes

The holomorphic disc $Z(\sigma) : D \to \mathbb{T}$ through $Z_i \in \mathbb{T}_{\mathbb{R}}$ at $\sigma_i \in \partial D$, with $\partial D \subset \mathbb{PT}_{\mathbb{R}}$ degree $k-1$ arise from action:

$$S_D[Z_i, h] = \int_D I_{AB} Z^A \bar{\partial} Z^B d\sigma + \oint_{\partial D} h(Z) d\zeta + \sum_i I_{AB} Z^A(\sigma_i) Z_i^B$$

Amplitudes are functionals $\mathcal{M}[h, \tilde{h}_i]$ of gravitational data:

- $h \in C^\infty(\mathbb{PT}_{\mathbb{R}}, \mathcal{O}(2))$for fully nonlinear SD part,
- $\tilde{h}_i \in C^\infty(\mathbb{PT}_{\mathbb{R}}, \mathcal{O}(-6))$, $i = 1, \ldots, k$, ASD perturbations.

Amplitude $\mathcal{M} :=$ Einstein-Hilbert action S_{EH} of 'bulk metric'.
Evaluate perturbatively:

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Evaluate perturbatively:

- At $k = 1$, $h = h_1 + h_2$ and we find to second order

$$\mathcal{M}[h_1, h_2, \tilde{h}] = \delta^3 S_{\mathrm{EH}}[\tilde{h}, h_1, h_2] = \int D^3 Z \, \tilde{h}(Z) \delta^2 S_D[Z, h]$$

$$= \int D^3 Z \, \tilde{h}(Z) \{h_1(Z) h_2(Z)\}$$

- At higher k disk action embeds into Einstein Hilbert action.
- Full gravity amplitudes invariant under hol. Poisson diffeos.

Conclusions and open problems

- Rigidity of conformally-flat SD split signature Poincaré-Einstein metrics with $\mathscr{I} = S^2 \times S^1/\mathbb{Z}_2$.
- Have construction for split signature SD Poincaé-Einstein metrics on $S^2 \times S^2$ with $\mathscr{I} \simeq S^2 \times S^1$ depending on smooth sections h of $\mathcal{O}(2)$ over $\mathbb{RP}^3$.
- Similar results at $\Lambda = 0$ where it is possible to reconstruct h from data at $\mathscr{I}$, M. 2212.10895, [also 2103.16984 w/ Sharma, Adamo].

Open questions:

- characterize Zollfrei conformal structures on $S^2 \times S^1$ with Poincaré-Einstein extensions over $S^2 \times S^2$.
- Understand extent to which hidden symmetries extend beyond SD sector.
- Express ASD perturbations nonperturbatively.

The end

Thank You

Examples for $\Lambda = 0$: Gibbons-Hawking

- Let $Z^A = (\lambda_\alpha, \mu^\alpha)$, $\alpha, \beta = 0, 1$; set $\varepsilon_{\alpha\beta} = \varepsilon_{[\alpha\beta]}$ and

$$\theta = \lambda_\alpha d\lambda_\beta \varepsilon^{\alpha\beta}\,, \qquad \{f, g\} = \varepsilon^{\alpha\beta} \frac{\partial f}{\partial \mu^\alpha} \frac{\partial g}{\partial \mu^\beta}\,,$$

- **Defn of** $\mathbb{PT}_\mathbb{R}$: λ_α is real and for $\mu^{\dot\alpha} = u^\alpha + iv^\alpha$, take

$$v^\alpha = \lambda^\alpha \dot{h}\,, \qquad h = (u^\alpha \lambda_\alpha, \lambda_\alpha)\,.$$

- Use λ_α as homogeneous coordinates on the hol. disks, expressed as graphs by

$$\mu^\alpha = x^{\alpha\beta}\lambda_\beta + (t + g(x, \lambda))\lambda^\alpha\,, \qquad x^{\alpha\beta} = x^{(\alpha\beta)}\,.$$

  where

$$g(x^{\alpha\beta}, \lambda) = \oint \frac{\lambda_0 - i\lambda_1}{\lambda'_0 - i\lambda'_1} \frac{1}{\langle \lambda\lambda' \rangle} \dot{h}((x^{\alpha\beta}\lambda'_\alpha\lambda'_\beta, \lambda'_\alpha) D\lambda'$$

- Gives split signature version of familiar metric

$$ds^2 = V d\mathbf{x}\cdot d\mathbf{x} + V^{-1}(dt + \omega)^2\,, \qquad dV =^* d\omega\,, \qquad V = \oint \ddot{h} D\lambda\,.$$

  But now V satisfies $2 + 1$ wave equation!