① **Goal**: compute abs relative (differential) inv of $G/\mathfrak{g}$-action

$\mathfrak{g}\subset\mathcal{D}(M)$, $0\neq f\in\mathcal{F}(M)$ is a relative inv if $X(f)=\lambda(X)f \quad \forall X\in\mathfrak{g}$.

Here $\lambda\in\mathfrak{g}^*\otimes\mathcal{F}(M)=:\mathrm{Hom}(\mathfrak{g},\mathcal{F}(M))$

BORIS KRUGLIKOV
talk at GRIEG conf. on
CARTAN GEOMETRY in ORSAY-24

- Hypersurface $\{f=0\}$ is invariant (in M)
- Action $L_{[X,Y]}f=[L_X,L_Y]f=\lambda([X,Y])f=X(\lambda(Y))-Y(\lambda(X))\Leftrightarrow d^1\lambda=0$
- Change of det. funct: $X(e^\mu f)=(X(\mu)+\lambda(X))e^\mu f \Leftrightarrow \lambda\mapsto\lambda+d^0\mu$

Chevalley-Eilenberg
$$0\to\mathcal{F}\xrightarrow{d^0}\mathfrak{g}^*\otimes\mathcal{F}\xrightarrow{d^1}\Lambda^2\mathfrak{g}^*\otimes\mathcal{F}\xrightarrow{d^2}\dots\rightsquigarrow H^1(\mathfrak{g},\mathcal{F})=\frac{\mathrm{Ker}\,d^1}{\mathrm{Im}\,d^0}$$
multipliers

What's known abt this cohomology?

- Whitehead lemma: $\mathfrak{g}$-ss, V-finite dim module $\Rightarrow H^1(\mathfrak{g},V)=0$
- Wrong if $\dim V=\infty$. Nothing known if $\dim\mathfrak{g}=\infty$, e.g. $\dim H^1(\mathfrak{g},V)\dots$
- Positive ex: $H^1(\cancel{\mathrm{Vect}(M)}\ \mathcal{D}(M), C^\infty(M))\simeq\mathbb{R}^{b_1(M)+1}$ [generators $\omega_1,\dots,\omega_{b_1(M)}$, div]
- Negative ex: $H^1(\cancel{\mathrm{Vect}(M)}, C^\infty(J^\infty(M)))$? Next que $H^1(\cancel{\mathrm{Vect}(M)}\ \mathcal{D}(M), C^\infty(J^\infty(J^2M)))$ {point equiv ODEs}

More subtle que: weight lattice $W\subset H^1(\mathfrak{g},\mathcal{F})$?

Idea: restrict the algebra of functions $C^\infty(M)\rightsquigarrow\mathcal{R}(M)$ or $\mathcal{M}(M)$.

We will begin with $\mathcal{O}(M)\overset{\text{sheaf}}{\rightsquigarrow}\mathcal{O}_M$, M-complex mnf (real analytic)

Recollection-I: line bundles $\pi:L\to M$. Loc. triv: atlas $\mathcal{U}=\{U_\alpha\}$:

$\pi^{-1}(U_\alpha)=U_\alpha\times\mathbb{C}$ w/trans. funct: $g_{\alpha\beta}:U_{\alpha\beta}\to\mathbb{C}^\times$

Čech
$$0\to\prod_\alpha\mathcal{O}^\times(U_\alpha)\xrightarrow{\delta^0}\prod_{\alpha\neq\beta}\mathcal{O}^\times(U_{\alpha\beta})\xrightarrow{\delta^1}\prod_{\alpha,\beta,\gamma}\mathcal{O}^\times(U_{\alpha\beta\gamma})\xrightarrow{\delta^2}\dots$$
$$\mu=\{\mu_\alpha\}\qquad g=\{g_{\alpha\beta}\}$$
$$(\delta^0\mu)_{\alpha\beta}=\mu_\alpha\mu_\beta^{-1}\qquad(\delta^1g)_{\alpha\beta\gamma}=g_{\alpha\gamma}g_{\beta\gamma}^{-1}g_{\beta\alpha}=g_{\alpha\gamma}g_{\gamma\beta}g_{\beta\alpha},\dots$$

$$\mathrm{Pic}(M)=\check{H}^1_{\mathcal{U}}(M,\mathcal{O}_M^\times)=(\lim_{\mathcal{U}})\frac{\mathrm{Ker}\,\delta^1}{\mathrm{Im}\,\delta^0}\qquad H^{0,1}_{\bar\partial}(M)\underset{\text{Kähl}}{\overset{\text{proj}}{\subset}}H^1(M,\mathbb{C})$$

$$0\to\mathbb{Z}\to\mathcal{O}_M\xrightarrow{\exp}\mathcal{O}_M^\times\to0\Rightarrow H^1(M,\mathcal{O}_M)\to H^1(M,\mathcal{O}_M^\times)\xrightarrow{c_1}H^2(M,\mathbb{Z})$$

Recollection-II: divisors D = global sect of the quot. sheaf $\mathcal{M}_M^\times/\mathcal{O}_M^\times$

Hypersurfaces loc. defined by functs: $\mathcal{U}=\{U_\alpha\}$, $f_\alpha\in\mathcal{M}(U_\alpha)$, $f_\beta/f_\alpha\in\mathcal{O}^\times(U_{\alpha\beta})\Rightarrow D\{f_\alpha=0\}$

$$\underbrace{H^0(M,\mathcal{M}_M^\times)}_{\sim}\to\underbrace{H^0(M,\mathcal{M}_M^\times/\mathcal{O}_M^\times)}_{\mathrm{Div}(M)}\to\underbrace{H^1(M,\mathcal{O}_M^\times)}_{\mathrm{Pic}(M)}\Leftarrow\left(0\to\mathcal{O}_M^\times\to\mathcal{M}_M^\times\to\mathcal{M}_M^\times/\mathcal{O}_M^\times\to0\right)$$

$\{f_\alpha\}\mapsto\{g_{\alpha\beta}=f_\beta/f_\alpha\}$: $L=[D]$

② **Def** $\mathfrak{g}\subset\mathcal{D}(M)$. A $\mathfrak{g}$-equivariant line bundle over M is a pair $(\pi: L\to M,\ \hat{\mathfrak{g}}\in\mathcal{D}(L))$ with $\hat{\mathfrak g}$ preserving the sheaf of linear in fib. funct on L (linearization) and $\pi_*\hat{\mathfrak g}=\mathfrak g$

Locally: $\pi^{-1}(U_\alpha)=U_\alpha\times\mathbb{C}$, $\hat{\mathfrak g}|_{\pi^{-1}(U_\alpha)}=\{X|_{U_\alpha}+\lambda_\alpha(X)u\partial_u : X\in\mathfrak g\}$

$\{\lambda_\alpha\}\mapsto[(\lambda_\alpha)]\in H^1(\mathfrak g, \mathcal O^{\#}_M(U_\alpha))$. Compatibility w/ trans. functions $\{g_{\alpha\beta}\}$

$$\lambda_\alpha(X)-\lambda_\beta(X)=X(g_{\alpha\beta})/g_{\alpha\beta}\quad\forall X\in\mathfrak g$$

Def A divisor $D=\{f_\alpha\}$ is $\mathfrak g$-inv. if for some $\lambda=\{\lambda_\alpha\}\in\prod_\alpha\mathfrak g^*\otimes\mathcal O(U_\alpha)$ we have $X(f_\alpha)=\lambda_\alpha(X)f_\alpha$ $\forall X\in\mathfrak g$

In the same way as in general context, we get $\boxed{j_{\mathfrak g}: \mathrm{Div}_{\mathfrak g}(M)\to \mathrm{Pic}_{\mathfrak g}(M)}$

Rk The grp $\mathrm{Pic}_G(M)$ is known (from before) in two cases:
- alg. grp G variety/scheme M
- compact grp G acting cont.

Double complex for $\mathfrak g$-equiv line bundles

$$\begin{tikzcd}
\prod_\alpha \mathcal O^\times_M(U_\alpha) \arrow[r, "d^0\log"] \arrow[d, "\delta^0"] & \prod_\alpha \mathfrak g^*\otimes\mathcal O_M(U_\alpha) \arrow[r, "d^1"] \arrow[d, "\delta^0"] & \prod_\alpha \Lambda^2\mathfrak g^*\otimes \mathcal O_M(U_\alpha) \arrow[r] \arrow[d, "\delta^0"] & \dots \\
\prod_{\alpha\beta}\mathcal O^\times_M(U_{\alpha\beta}) \arrow[r, "d^0\log"] \arrow[d, "\delta^1"] & \prod_{\alpha,\beta}\mathfrak g^*\otimes\mathcal O_M(U_{\alpha\beta}) \arrow[r, "d^1"] \arrow[d, "\delta^1"] & \prod_{\alpha,\beta}\Lambda^2\mathfrak g^*\otimes\mathcal O_M(U_{\alpha\beta}) \arrow[r] \arrow[d, "\delta^1"] & \dots \\
\prod_{\alpha,\beta,\gamma}\mathcal O^\times_M(U_{\alpha\beta\gamma}) \arrow[r] \arrow[d, "\delta^2"] & \prod_{\alpha,\beta,\gamma}\mathfrak g^*\otimes \mathcal O_M(U_{\alpha\beta\gamma}) \arrow[r, "d^1"] \arrow[d] & \dots & \\
\dots & \dots & &
\end{tikzcd}$$

$(g,\lambda)\in C^{0,1}\times C^{1,0}$

$d^1\lambda=0$

$\delta^0\lambda=\overset{\bar d^0}{d^0\log g}$

$\delta^1 g=0$

$$\mathrm{Tot}^r(C)=\prod_{p+q=r}C^{p,q}\qquad \partial^r=\sum_{p+q=r}\left(d^{p,q}+(-1)^p\delta^{p,q}\right):\mathrm{Tot}^r(C)\to\mathrm{Tot}^{r+1}(C)$$

Thm 1 ("upper ~~upper~~ bound") ①

$$0\to \frac{\ker(d^{0,1}\delta^{0,0})}{\ker(\delta^{0,0})\ker(d^{0,0})}\to \underline{H}^1(\mathrm{Tot}^\bullet C)\to\mathrm{Pic}_{\mathfrak g}(M)\to 0$$

Thm 2 ("upper ~~lower~~ bound") ②

$$\begin{tikzcd}
& \mathrm{Pic}_{\mathfrak g}(M) \arrow[dl, "\psi_1"'] \arrow[dr, "\psi_2"] & \\
\underbrace{\mathrm{Pic}(M)}_{\check H^1(M,\mathcal O^\times_M)} & & \underbrace{\mathcal{M}\ell t_{\mathfrak g}(M)}_{\varprojlim \prod_\alpha H^1(\mathfrak g,\mathcal O_M(U_\alpha))}
\end{tikzcd}$$

$\psi_1\times\psi_2$ inj

Mlt(g)

Consider the comm. diagram

$$\begin{tikzcd}
\mathrm{Div}_{\mathfrak g}(M) \arrow[r, "j_{\mathfrak g}"] \arrow[d, "\Phi_1"'] & \mathrm{Pic}_{\mathfrak g}(M) \arrow[d, "\psi_1"] \\
\mathrm{Div}(M) \arrow[r, "j"] & \mathrm{Pic}(M)
\end{tikzcd}$$

The condition $L=[D]\in\mathrm{Im}(\psi_1 j_{\mathfrak g})$ is restrictive

Thm 3 $L=[D]$, λ-wght of D $\Rightarrow$ $\hat{\mathfrak g}=\mathfrak g^\lambda$ is transversal

- The existence of relative invariant locally was related by [Fels-Olver '97] to preservation of dim($\mathfrak g$-orbits) in the lift
- [Anderson-Fels 2002] related this cond (called transversality) to charact. of inv. sections

③ Mumford in GIT book proves that if

- G is algebraic
- M is projective/scheme $\Longrightarrow$ then $\forall\, L \in Pic(M)$ $\exists$ at most one lift (linearization) $\Leftrightarrow Pic_G(M) \hookrightarrow Pic(M)$
- $\nexists\, G \twoheadrightarrow GL(1)$

The last condition holds for ss grps but fails for solvable..

For Lie alg $\mathfrak{g}$ and general M the claim fails. For instance, $\mathfrak{g} = sl(2,\mathbb{C})$, $M = \mathbb{C}$ realization proj $\langle \partial_z, z\partial_z, z^2\partial_z\rangle$ $Pic_{\mathfrak{g}}(M) = \mathbb{C}$, $Pic(M) = 0$

Indeed, the lifts are $\langle \partial_z, z\partial_z + A u\partial_u, z^2\partial_z + 2Azu\partial_u\rangle = \hat{\mathfrak{g}}$

Our version of his result is related to the property of stabilizer $\mathfrak{g}_p = \{v \in \mathfrak{g} : v(p) = 0\}$, $p \in M$.

Thm 4. Let $\mathfrak{g}$ be a transitive Lie alg. of vector fields on M with $H^1_{dR}(M) = 0$. If $\nexists\, \mathfrak{g}_p \twoheadrightarrow gl(1,\mathbb{C})$ for $p \in M$, then $Pic_{\mathfrak{g}}(M) \hookrightarrow Pic(M)$.

Ex 1 $M = \mathbb{C}P^1$, $\mathfrak{g} = aff(1) = \langle X, Y\rangle$ $[X,Y] = X$

$U_0 = \mathbb{C}^1(z)$, $U_\infty = \mathbb{C}^1(w)$, $w = \frac{1}{z}$ :: $X|_{U_0} = \partial_z$, $Y|_{U_0} = z\partial_z$; $X|_{U_\infty} = -w^2\partial_w$, $Y|_{U_\infty} = -w\partial_w$

$\lambda = \{\lambda_0, \lambda_\infty\}$: $\lambda_0(X) = 0$, $\lambda_0(Y) = A$; $\lambda_\infty(X) = Bw$, $\lambda_\infty(Y) = C$

Compatibility: $\lambda_\infty(X) - \lambda_0(X) = X(\log(g_{0\infty})) \Rightarrow Bw\, g_{0\infty} = -w^2\partial_w(g_{0\infty})$

$\lambda_\infty(Y) - \lambda_0(Y) = Y(\log(g_{0\infty})) \Rightarrow (C-A)\, g_{0\infty} = -w\partial_w(g_{0\infty})$

$B = C - A$, $g_{0\infty} = w^{-B} = z^B \Rightarrow B \in \mathbb{Z}$

We conclude $Pic_{\mathfrak{g}}(M) = \mathbb{C}\times\mathbb{Z}$, $Pic(M) = \mathbb{Z}$

Ex 2 For $M = \mathbb{C}P^n$, $\mathfrak{g} = sl(n+1,\mathbb{C})$ we similarly get

$Pic_{\mathfrak{g}}(M) = \mathbb{Z} = Pic(M)$

Ex 3. For $M = \mathbb{C}P^n$, $G = PGL(n+1)$: $Pic_G(M) = (n+1)\mathbb{Z} \overset{n+1}{\subset} \mathbb{Z} = Pic(M)$

Note that $SL(n+1)$ allows linearization (lift) to any $\mathcal{O}(k)$, but $PGL(n+1)$ only to $\mathbb{Z}$-powers of ~~those~~ ~~proportional to~~ $K^* = \Lambda^n T^*\mathbb{C}P \simeq \mathcal{O}_{\mathbb{C}P^n}(-n-1)$ because $Z(GL(n+1)) = \mathbb{Z}_{n+1}$

Thm 5. Let D be a $\hat{\mathfrak{g}}$ inv. polynomial divisor on affine bundle $E \to M$. Then $\exists$ line bundle $L \xrightarrow{\pi} M$, $L \in \Psi_1(Pic_{\mathfrak{g}}(M))$ s.t. $[D] = \pi^* L$

Corollary Let $\mathfrak{g}^{(k)} \in \mathcal{D}(J^k)$ be prolong of a Lie alg $\mathfrak{g} \subset \mathcal{D}(J^0)$. If D is a $\mathfrak{g}^{(k)}$-inv. polynomial divisor in fibers of $\pi_{k,1} : J^k \to J^1$ then $[D] = \pi^*_{k,1} L$ for $L \to J^1$.