Quasi Einstein structures

Maciej Dunajski

DAMTP, University of Cambridge

- M. Dunajski, James Lucietti (2023) Intrinsic rigidity of extremal horizons. `arXiv:2306.17512`
- Alex Colling, M. Dunajski, Hair Kunduri, James Lucietti (2024) New quasi-Einstein metrics on a two–sphere. `arXiv:2403.*****`

Quasi–Einstein equations

(M, g) closed n–dimensional Riemannian manifold, $X \in \mathfrak{X}(M)$.

$$\text{Ric}(g) = \frac{1}{m} X^\flat \otimes X^\flat - \frac{1}{2}\mathcal{L}_X g + \lambda g, \quad m \neq 0, \lambda \text{ constants.} \quad (QEE)$$

- $m = 2$. (M, g) spatial cross–section of extremal black hole horizon with cosmological constant λ (Lewandowski–Pawłowski 2003, Kunduri–Lucietti 2009, ...)
- $m = 1 - n, \lambda = 0$. Levi–Civita connection of (M, g) projectivelly equivalent to a connection with skew Ricci (local results: Randall 2014, Nurowski–Randall 2016).
- $n = 3, m = 1, \lambda = 0$. (M, g) initial data for a static solution to Lorentzian Einstein equations in $(1, 3)$ signature (Bartnik–Tod 2005).
- $m \in \mathbb{N}$. Warped product Einstein metric on $M \times F$. (Kim–Kim 2003)

$$G = g + e^{-\frac{2f}{m}} g_F, \quad X^\flat = df, \quad \text{Ric}(g_F) = \mu g_F, \quad \dim(F) = m.$$

- $m = \infty$. Ricci solitons. (Hamilton 1988).

Extreme Kerr horizon

Example: $M = S^2, m = 2, \lambda = 0$.

$$g = \frac{a^2(1+x^2)\mathrm{d}x^2}{1-x^2} + \frac{4a^2(1-x^2)\mathrm{d}\phi^2}{1+x^2}, \quad \text{(Kerr)}$$

$$X^\flat = \frac{K^\flat - \mathrm{d}\Gamma}{\Gamma}, \quad \Gamma = \frac{1+x^2}{2}, \quad \text{where} \quad K = \frac{1}{2a^2}\partial_\phi$$

where $a > 0$ is a constant, and $-1 \leq x \leq 1, \phi \in [0, 2\pi]$.

- Question: Is (Kerr) the unique solution to QEE with $(m = 2, \lambda = 0)$ on a two–sphere?
- Lewandowski–Pawłowski (2003), Kunduri–Lucietti (2009). Yes, if there exists a Killing vector preserving $X^\flat$.
- Chruściel–Szybka–Tod (2018). Yes, in the neighbourhood of (Kerr) in the space of solutions to QEE.
- MD–Lucietti (2023): Yes, with no additional assumptions (global rigidity of extremal Kerr horizon) , and more is true.

Rigidity of extreme horizons

Theorem (D–Lucietti 2023): Let (M, g) be an n-dimensional compact Riemannian manifold without boundary admitting a non-gradient vector field X such that QEE holds with $m = 2$. Then (M, g) admits a Killing vector field K. Furthermore, if either (i) $\lambda \leq 0$, or (ii) $n = 2$ and λ is arbitrary, then $[K, X] = 0$.

- Corollary 1: The extremal Kerr horizon (possibly with cosmological constant) is the unique solution to $m = 2$ QEE on $M = S^2$.
- Corollary 2: Any non-trivial vacuum near-horizon geometry with $\lambda \leq 0$ and compact cross-sections admits an isometric action of $SO(2, 1)$ with 3–dim orbits. $(n + 2)$-dimensional Lorentzian metric

$$\mathbf{g} = r^2 \mathrm{d}v^2 + 2\mathrm{d}v\mathrm{d}r + 2rX^\flat \odot \mathrm{d}v + g, \quad \text{where } \; F = \frac{1}{2}|X|^2 - \mathrm{div}(X^\flat) + \lambda.$$

- Proof: *principal eigenvalues of elliptic operators* (global), remarkable tensor identity (local, valid if $m = 2$), integration by parts (global).
- Alex Colling: Theorem valid for any λ with $n > 2$.

Proof. Step one: principal eigenfunction

Lemma. Given any vector field X on a compact Riemannian manifold (M, g) there exists (a unique up to scale) smooth function $\Gamma > 0$ such that $\text{div}(K) = 0$, where

$$K^\flat := \Gamma X^\flat + \text{d}\Gamma.$$

- Elliptic differential operator $L\psi := -\text{div}(d\psi + X^\flat\psi)$. Krein-Rutman theorem: Mcompact: there exists a principal eigenvalue $\mu \in \mathbb{R}$ less than or equal to the real part of any other eigenvalue, whose associated eigenfunction ψ is everywhere positive and unique up to scale.
- M closed: Integrate $L\psi = \mu\psi$ by parts, deduce $\mu = 0$.
- K is divergence–free, so a candidate for a Killing vector. Have not used QEE. Now use it!

Proof. Step two: tensor identity

Proposition: Let $K^\flat := \Gamma X^\flat + (\mathrm{d}\Gamma)$. For any solution to QEE with $m = 2$ the following identity holds

$$
\begin{aligned}
\nabla_{(a}K_{b)}\nabla^a K^b &= \nabla^a\left(K^b\nabla_{(a}K_{b)} - \tfrac{1}{2}K_a\Delta\Gamma - \tfrac{1}{2}K_a\nabla_b K^b - \lambda\Gamma K_a\right)\\
&+ \nabla_b K^b\left(-\frac{1}{2\Gamma}|K|^2 + \frac{1}{2}\Delta\Gamma + \frac{1}{2}\nabla_b K^b + \frac{1}{2\Gamma}K^b\nabla_b\Gamma + \lambda\Gamma\right)
\end{aligned}
$$

- Proof: Substitute $X^\flat = \Gamma^{-1}(K^\flat - \mathrm{d}\Gamma)$ to QEE. Calculate. Check. Correct errors. Check again. Does it also work for $m \neq 2$? No. Did it *really* work for $m = 2$ (so many unexplainable cancellations)? Check again. Yes. Remarkable ..
- Take Γ to be the principal eigenfunction, so that $\nabla_b K^b = 0$. Stokes theorem:

  $$\int_M |\mathcal{L}_K g|^2 \,\mathrm{vol}_M = \int_M \mathrm{div}(\dots)\mathrm{vol}_M = 0.$$

  (M, g) Riemannian, so that $\mathcal{L}_K g = 0$.
- More work: $\mathcal{L}_K X = 0$.

$\mathrm{Ric}(g) = \frac{1}{m} X^\flat \otimes X^\flat - \frac{1}{2}\mathcal{L}_X g + \lambda g$ WITH $m \neq 2$

- Focus on M compact without boundary, and $n = 2$.
- Take the trace of QEE. Use Gauss–Bonnet: Let $\mathrm{g}_M = \mathrm{genus}(M)$. Then
  - If $m > 0$ and $\lambda > 0$, then $\mathrm{g}_M = 0$.
  - If $m > 0$ and $\lambda = 0$, then $\mathrm{g}_M \leq 1$ ($= 1$ iff (M, g) is the flat torus).
  - If $m < 0$ and $\lambda < 0$, then $\mathrm{g}_M > 1$.
  - If $m < 0$ and $\lambda = 0$, then $\mathrm{g}_M \geq 1$ ($= 1$ iff (M, g) is the flat torus).
- Dobkowski-Ryłko, Kamiński, Lewandowski, Szereszewski (2018): If $m = 2$ and $\mathrm{g}_M > 0$, then $X \equiv 0$. Colling (2024): If $\mathrm{g}_M > 0$ then $X \equiv 0$.
- ... so focus on $\mathrm{g}_M = 0$. Find all regular solutions with a Killing vector. First local, and then global which extend to S^2 or $\mathbb{RP}^2$.

Theorem (Colling, D, Kunduri, Lucietti 2024): Let (g, X) be a solution to the $m-$quasi–Einstein equation on a two–dimensional surface M with $dX^\flat$ not identically zero, and a $U(1)$ isometric action. Then

- Locally there exist coords. (x, ϕ), and a function $B = B(x)$ s. t.

$$
\begin{aligned}
g &= B^{-1}dx^2 + Bd\phi^2, \quad X^\flat = \frac{-m}{x^2+1}\Big(x\mathrm{d}x - B\mathrm{d}\phi\Big), \quad (B)\\
B &= \begin{cases} bx(x^2+1)^{-m/2} + c(x^2+1)^{-m/2}F(x) - \frac{\lambda(x^2+1)}{m+1}, & m \neq -1\\ x\Big(b - \lambda\mathrm{arcsinh}(x)\Big)\sqrt{x^2+1}) + c(x^2+1), & m = -1. \end{cases}
\end{aligned}
$$

where b, c are constants and $F(x) \equiv {}_2F_1\Big(-\frac{1}{2}, -\frac{m}{2}, \frac{1}{2}, -x^2\Big)$ is the hyper–geometric function.

- If $b = 0$ and

	$\lambda = 0$	$\lambda > 0$	$\lambda < 0$
$m > 0$	$c > 0$	$c > \frac{\lambda}{m+1}$	$c > \frac{\lvert\lambda\rvert}{m+1} c_0$
$m \in (-1, 0)$	-	$c > \frac{\lambda}{m+1}$	-
$m = -1$	-	$c > 0$	-
$m < -1$	-	$c \in \left(\frac{\lambda}{m+1}, 0\right)$	-

  where

$$c_0 = \min_{x > x_0} \frac{(x^2+1)^{\frac{m}{2}+1}}{\lvert F(x)\rvert}$$

  and x_0 is the unique positive zero of F, then (B) smoothly extends to S^2.

- Conversely all solutions to QEE on S^2 with a $U(1)$ isometric action arise from (B) with $b = 0$, together with the restrictions on c given above.

Some 'ingredients' of the proof

- Lemma: Let (M, g, X) be a quasi-Einstein manifold of dimension n. In harmonic coordinates the components of X and g are real analytic. (Proof: Deturck–Kazdan elliptic theory).
- Proposition: Let (g, X) be a solution to the quasi-Einstein equations on a two-dimensional connected surface M admitting a Killing vector K. Then either $[K, X] = 0$ or g has constant curvature. (Proof: Nurowski–Randall prolongation$+U(1)$ adapted coordinates).
- Lemma: (not ours - folklore): The metric

$$g = B^{-1}dx^2 + Bd\phi^2, \quad \text{where} \quad B = B(x)$$

extends to a smooth metric on S^2 if and only if there exist adjacent simple zeros $x_1 < x_2$ of B such that $B > 0$ for all $x_1 < x < x_2$ and $B'(x_1) = -B'(x_2)$ where $\phi \sim \phi + p$ is periodically identified with period $p = 4\pi/|B'(x_i)|$.

Kähler potential

Propositon: Locally there exist complex coordinates $(\zeta, \bar{\zeta})$ and a function f on M such that

$$g = 4 f_{\zeta\bar{\zeta}}\, \mathrm{d}\zeta \mathrm{d}\bar{\zeta}, \quad X^\flat = -m f_{\zeta\bar{\zeta}} (\mathrm{d}\zeta / f_{\bar{\zeta}} + \mathrm{d}\bar{\zeta}/f_\zeta),$$

and QEE reduces to a single 4th order PDE

$$\frac{2}{m}(f_\zeta f_{\bar{\zeta}})^2 (f_{\zeta\zeta\bar{\zeta}\bar{\zeta}} f_{\zeta\bar{\zeta}} - f_{\zeta\zeta\bar{\zeta}} f_{\zeta\bar{\zeta}\bar{\zeta}}) + \frac{4\lambda}{m}(f_{\zeta\bar{\zeta}})^3 (f_\zeta)^2 (f_{\bar{\zeta}})^2$$
$$-(f_{\zeta\bar{\zeta}})^3 (f_{\bar{\zeta}\bar{\zeta}}(f_\zeta)^2 + f_{\zeta\zeta}(f_{\bar{\zeta}})^2) + (f_{\zeta\bar{\zeta}})^2 (f_\zeta (f_{\bar{\zeta}})^2 f_{\zeta\zeta\bar{\zeta}} + f_{\bar{\zeta}}(f_\zeta)^2 f_{\zeta\bar{\zeta}\bar{\zeta}})$$
$$+2(f_{\zeta\bar{\zeta}})^4 f_\zeta f_{\bar{\zeta}} = 0.$$

Proof of the main theorem: Use Propositions Kähler and Inheritance ($U(1)$ isometry extends to X) to deduce (B). Solve the QEE for $B = B(x)$. Use the Folklore Lemma and hyper–geometric identities (thanks to Jan Deredziński!) to argue that B is even, and show that regular sols exist for all ranges of m and λ allowed by Gauss-Bonnet.

$m = -1$ AND PROJECTIVE METRISABILITY

- A projective structure $[\nabla]$ is an equivalence class of affine connections the same unparametrised geodesics. A projective structure is called
  - *metrisable* (M) if it contains a Levi–Civita connection.
  - *skew* (S) if it contains a connection with totally skew Ricci tensor.
  - Bryant-D-Eastwood (2009). Necessary and sufficient conditions for (M)
  - Randall (2014), Kryński (2014). Some necessary conditions for (S).
  - Question (open): Find all projective structures which are (M&S).
- Define an affine connection $D \equiv \nabla - p\, X^\flat \otimes \text{Id} - q\, \text{Id} \otimes X^\flat$.
  Proposition: QEE on a surface are equivalent to the flatness of D iff $m = -1, p = -\frac{1}{2}, q = 1$. (Proof: calculate).
- Milnor (1954): If closed orientable surface M admits a flat connection, then M is diffeomorphic to torus.
- Corollary 1: (use Milnor+Gauss–Bonnet+ Proposition): The only quasi–Einstein structure on a closed orientable surface with $m = -1, \lambda = 0$ is the flat torus.
- This is trumped by Colling's result (no–nontrivial $\mathrm{g}_M > 1$ QEE)
- ... but implies Corollary 2: The only compact orientable projective structure which is (M&S) is the flat torus.

Summary

- Quasi–Einstein structure (g, X) on a closed manifold M
  - If $m = 2$ then a Killing vector must exist, and preserve X. Rigidity of extreme Kerr horizon.
  - If $n = 2, m \neq 2$ we *assume* that a Killing vector exists, and find all local solutions, and solutions which extend to $M = S^2$.
  - No non–trivial solutions if genus$(M) > 0$.
  - QEE with $n = 2$ interpreted as a flat affine connection iff $m = -1, \lambda = 0$.
- Why talk about this at the Cartan meeting?
  - ... Paris is nice, and I have not seen Ben, Gil, Ian, Paweł, . . . for a decade (?)
  - A mixture of 'local' and 'global'. To do global, must do local first. Prolongation leads to a bundle with a non–linear connection.

Thank You