Dancing pairs, rolling balls and the Cartan-Engel distribution

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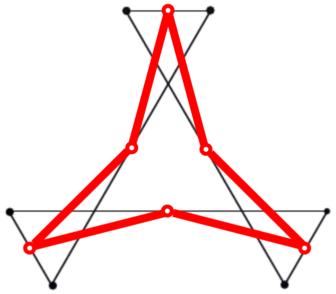
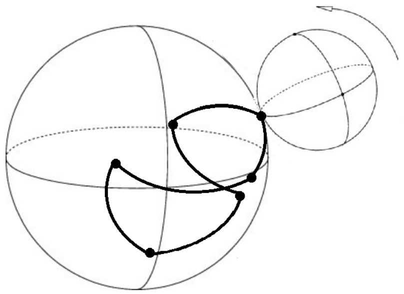
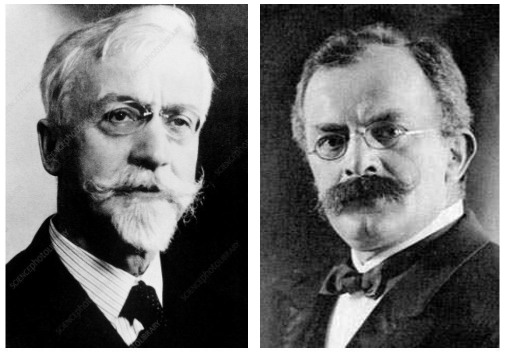
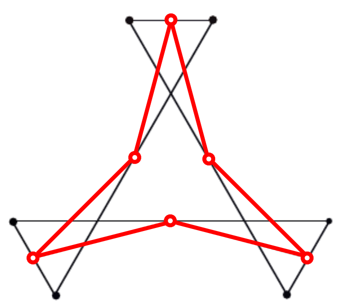


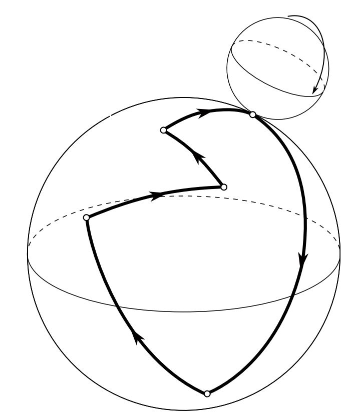




Theorem (main): there is a 1:1 correspondence

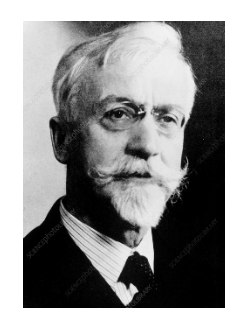
{ Dancing pairs of planar polygons } ⟷ { Spherical polygons with trivial rolling monodromy }

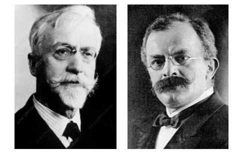




Proof:

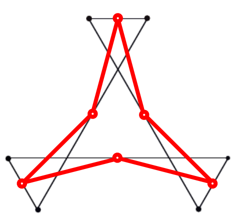
{ Closed piece-wise rigid Horizontal curves of the Cartan-Engel distribution }



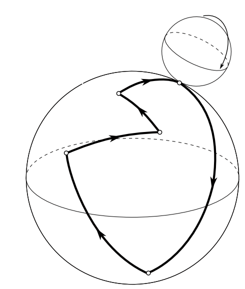


$\widetilde{Q}^{\texttt{oct}}$

2:1   2:1



$Q^{\texttt{dance}} \subset Q^{\texttt{oct}}$ $Q^{\texttt{roll}}$

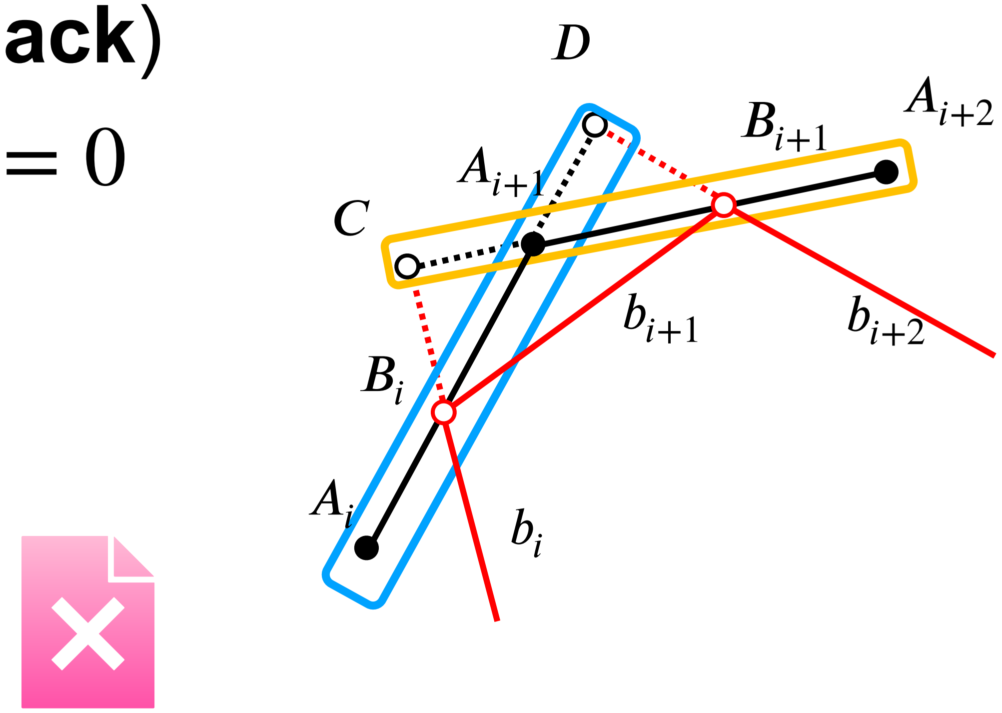


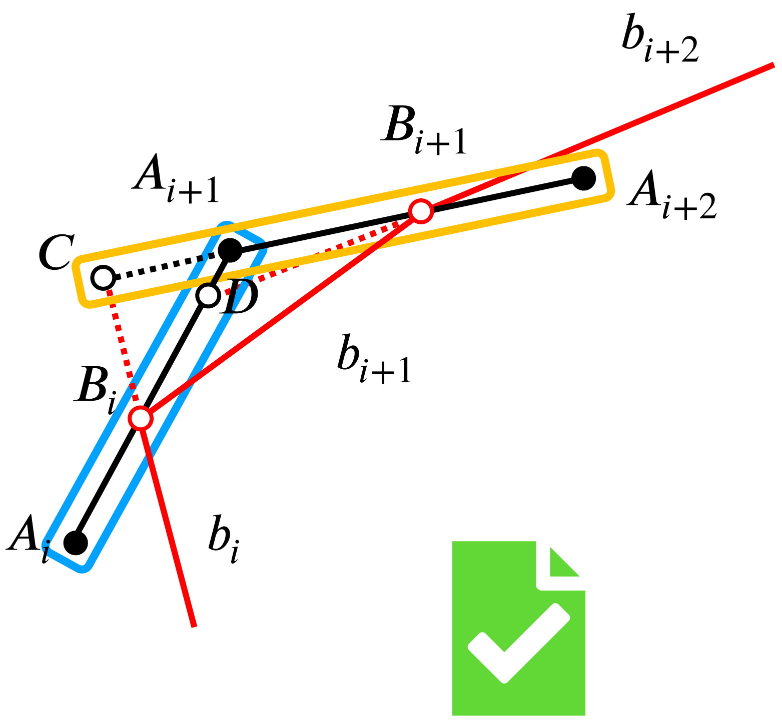
Dancing pair: is a pair of polygons in $\mathbb{RP}^2$, with vertices $A_1,\ A_2,\ \ldots,\ A_n$ and edges $b_1,\ b_2,\ \ldots,\ b_n$, such that for all i

(1) $b_i b_{i+1} \in A_i A_{i+1}$ (**red** is inscribed in **black**)

(2) $\left[A_{i+1}, B_i, A_i, D\right] + \left[A_{i+1}, B_{i+1}, A_{i+2}, C\right] = 0$

$$[x_1, x_2, x_3, x_4] := \frac{(x_1 - x_3)(x_2 - x_4)}{(x_1 - x_4)(x_2 - x_3)}$$





Non degeneracy conditions:

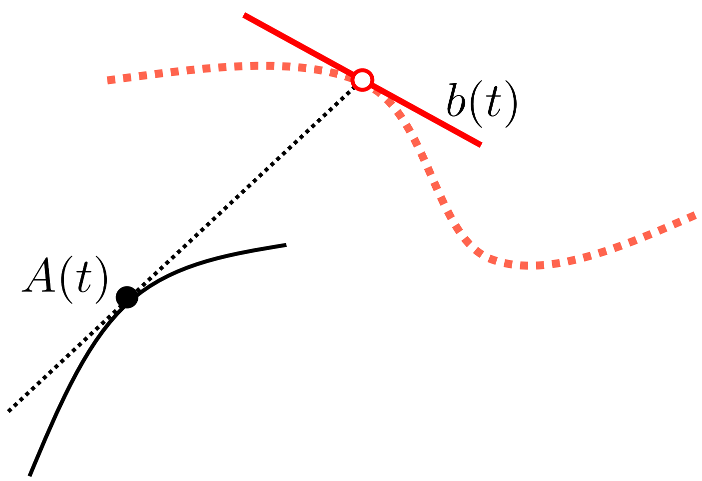
(1) No 3 consecutive A's are colinear.

(2) No 3 consecutive b's are concurrent.

(3) $A_i \notin b_i$, for all i.

Remark: dancing pairs of polygons = discrete version of ‘dancing’ pt-line pairs:

(1) $A(t)$ always moves towards the “turning pt” of $b(t)$ (‘ice skate’)



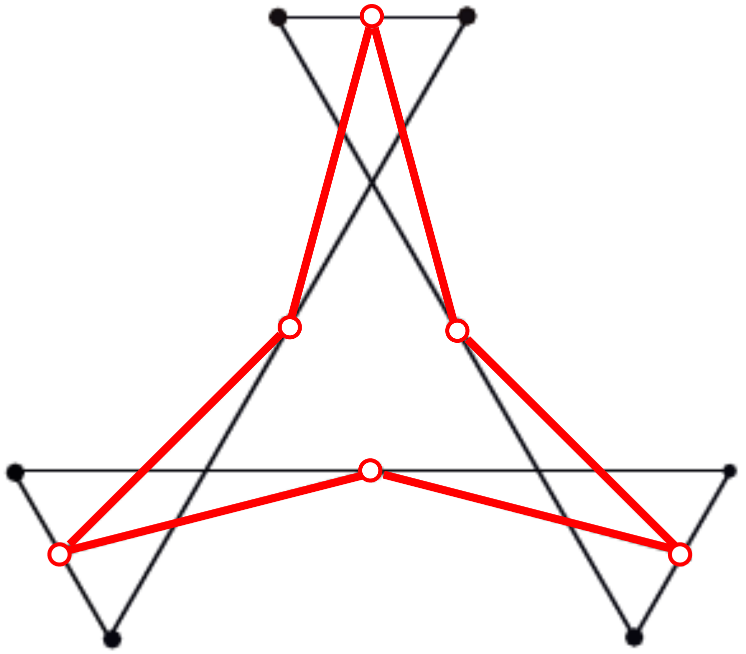
⇔ null curves of a **conformal metric of signature (2,2)** on

$$M^4 = \{(A, b) | A \notin b\} \subset \mathbb{R}P^2 \times \left(\mathbb{R}P^2\right)^*$$

(2) The tangent SD 2-plane along the curve $(A(t), b(t))$ in M is **parallel**

(a ‘half-geodesics’)

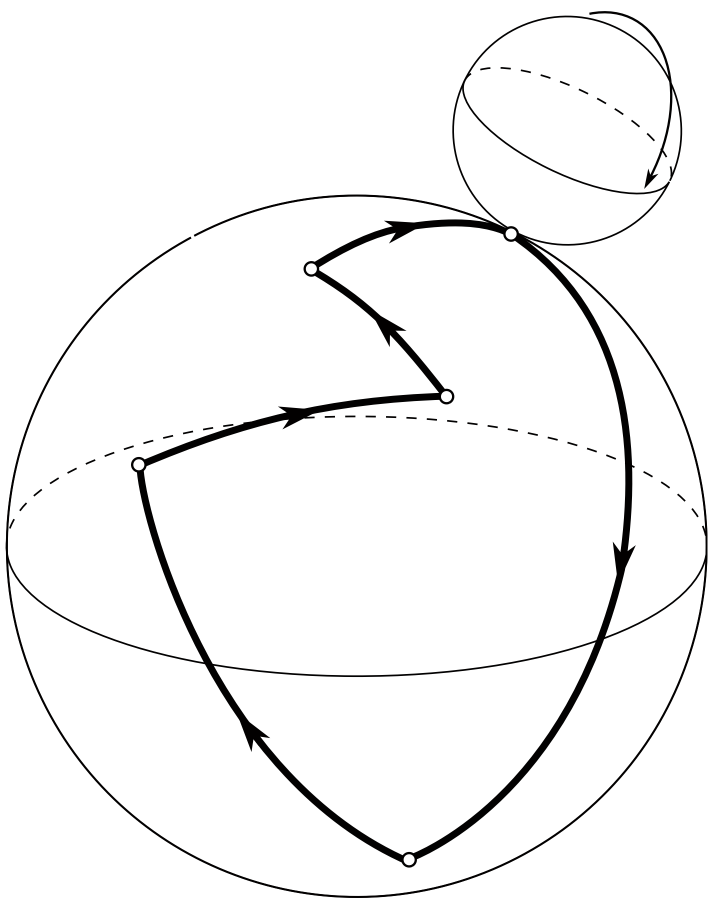
Theorem: there are dancing pairs of closed n gons iff $n \geq 6$.



$n = 6$

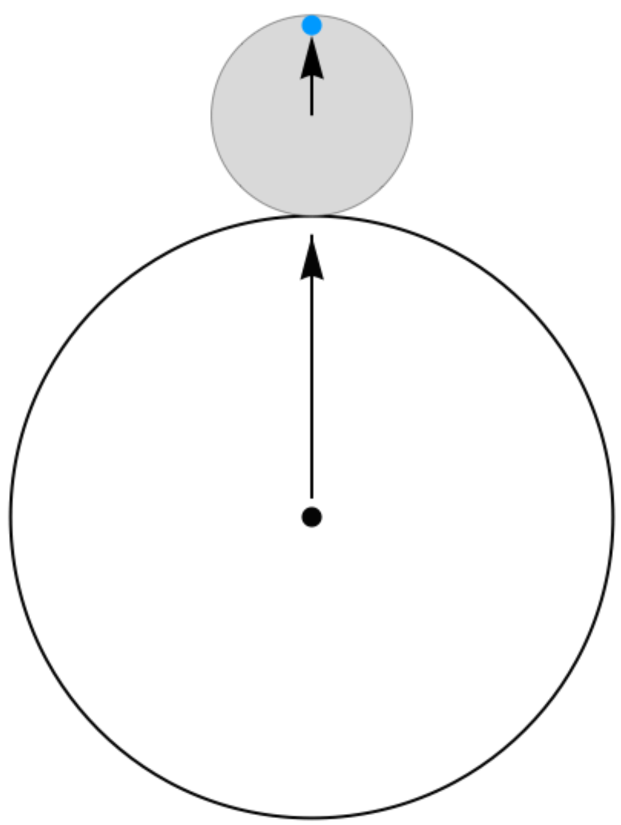
Proof: via rolling balls

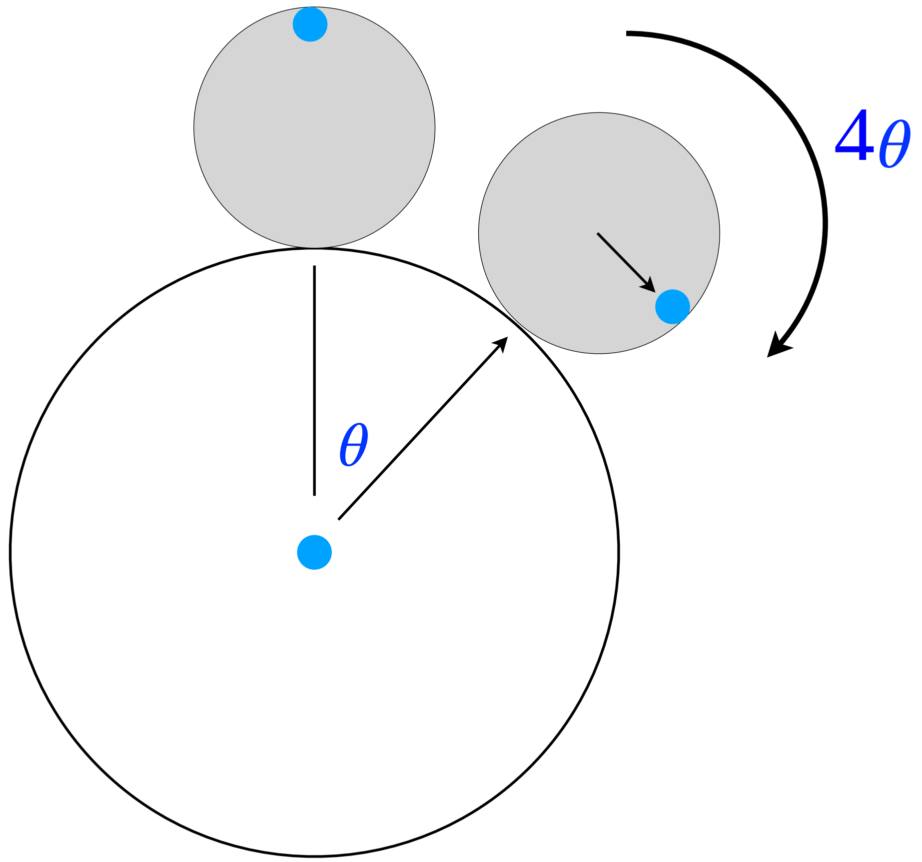
Rolling balls: a sphere of radius 1 is rolling without sliding and twisting along a closed polygon on a sphere of radius 3



The rolling ball defines a path in SO_3, the **rolling monodromy**

Rolling monodromy (3:1 ratio)

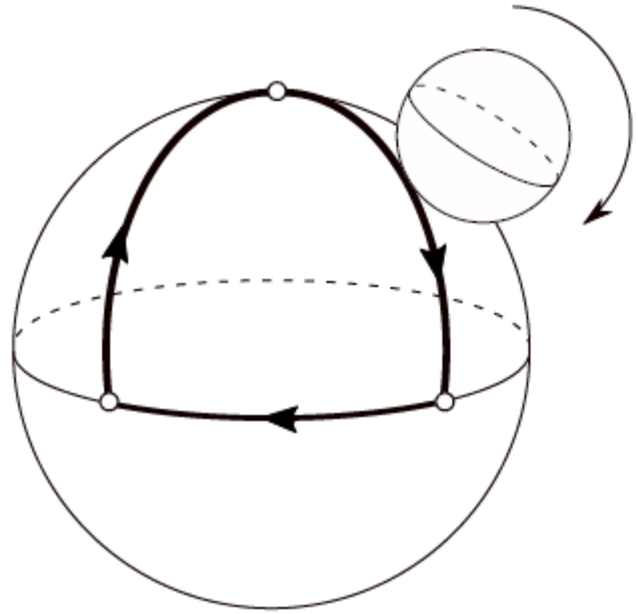




Definition: the rolling monodromy is **trivial** if the corresponding path in SO_3 is **closed** and **contractible.**

Equivalently: the lifted path in S^3 is **closed.**

Example: a triangular 'octant':



Each edge is ¼ of a great circle of the big sphere

$\Longrightarrow$ small sphere makes 1 full turn going along each edge

$\Longrightarrow$ lifted monodromy for each edge is -1

$\Longrightarrow$ monodromy of the triangle is $(-1)^3 = -1$

$\Longrightarrow$ monodromy of going **twice** around the triangle is trivial.

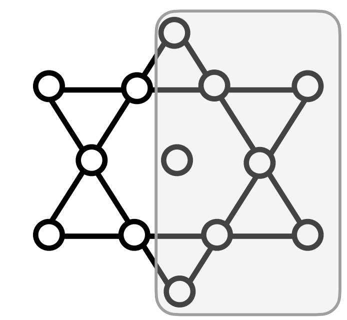
Cartan-Engel distribution (1893): a “flat” 2-plane dist $D \subset TQ$, max non-integrable, on a 5-mnfld Q

The “octonionic” model:

$Q^{\mathrm{oct}} \subset \mathbb{P}(\mathrm{Im}(\mathbb{O}))$, $D = \{\zeta d\zeta = 0\}$, $\mathrm{G}_2 = \mathrm{Aut}(\mathbb{O})$ - invariant

Projectivized null cone

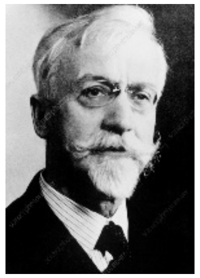
split oct.



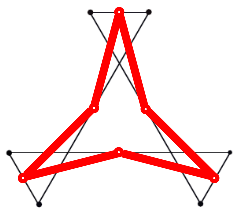
Theorem (Cartan, 1910):

(1) The (loc) symmetry gp of this 235 dist is G_2 (a 14-dim non-cpct simple Lie gp), max-dim possible for a 235 dist.

(2) All “flat” 235 dist (with G_2 symmetry) are loc diffeo.



(3) Submax symmetry for 235 dist: 7-dim.



$$\left\{ \begin{array}{c} \text{Dancing pairs of} \\ \text{planar polygons} \end{array} \right\} \longleftrightarrow \left\{ \begin{array}{c} \text{Closed piece-wise rigid} \\ \text{Horizontal curves of the} \\ \text{Cartan-Engel distribution} \end{array} \right\}$$

$$Q^{\mathtt{dance}} = \{\mathbf{b}\cdot\mathbf{A} = 1\} \subset \mathbb{R}^3 \times (\mathbb{R}^3)^*$$

$D := \{d\mathbf{b} = \mathbf{A} \times d\mathbf{A}\} \subset TQ^{\mathtt{dance}}$, the **dancing distribution,** is a Cartan-Engel 235-distribution.

(Pf: 8-dim symm $\Longrightarrow$ 14-dim symm)

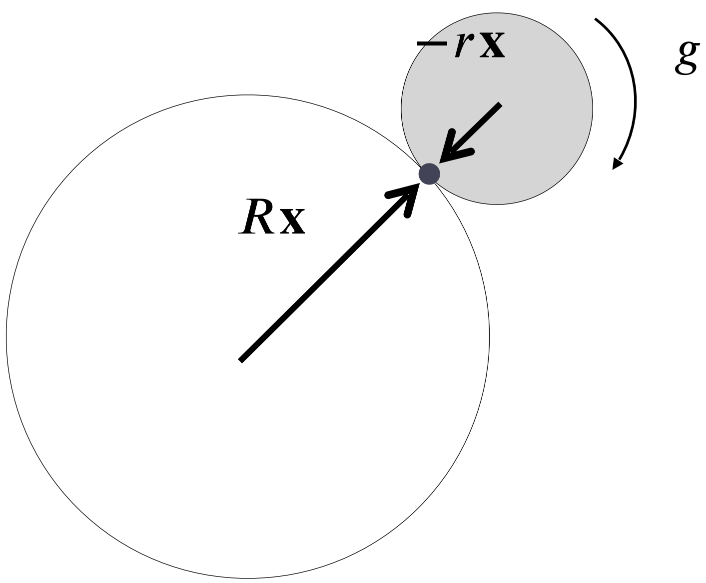
GB, L Hernandez, P Nurowski (2018), *The dancing metric, G_2-symmetry and projective rolling*, Trans. Amer. Math. Soc. **370(6)**

Theorem: a pair of polygons with vertices $A_1, \ldots, A_n \in \mathbb{RP}^2$ and edges $b_1, b_2, \ldots, b_n \in (\mathbb{RP}^2)^*$ is *dancing* iff there are homogeneous coordinates $\mathbf{A}_1, \ldots, \mathbf{A}_n \in \mathbb{R}^3, \mathbf{b}_1, \ldots, \mathbf{b}_n \in (\mathbb{R}^3)^*$, such that $(\mathbf{A}_1, \mathbf{b}_1), \ldots, (\mathbf{A}_n, \mathbf{b}_n)$ are the vertices of a horizontal polygon in $Q^{\mathtt{dance}}$.

$$
\begin{array}{ccc}
(\mathbf{A}, \mathbf{b}) & & Q^{\mathtt{dance}} \subset \mathbb{R}^3 \times (\mathbb{R}^3)^* \\
\downarrow & & \downarrow \\
(A, b) & & \mathbb{RP}^2 \times (\mathbb{RP}^2)^*
\end{array}
$$

{ Rolling balls trajectories } $\longleftrightarrow$ { Horizontal curves of the **Rolling distribution** }

$Q^{\texttt{roll}} = S^2 \times \mathrm{SO}_3$= configuration space for rolling balls



$$\begin{array}{ccc} \mathrm{SO}_3 & \longrightarrow & Q^{\texttt{roll}} \\ & & \downarrow \\ & & S^2 \end{array}$$

The **rolling distribution:** $D \subset TQ^{\texttt{roll}}$, 235-dist if $\rho = R/r \neq 1$,

$$\begin{cases} (\rho + 1)\mathbf{x}' = \boldsymbol{\omega} \times \mathbf{x}, \\ \boldsymbol{\omega} \cdot \mathbf{x} = 0 \end{cases}$$

$\mathbf{x} \in S^2, \ \ \boldsymbol{\omega} = g^{-1}g' \in \mathbb{R}^3 \simeq \mathfrak{so}_3$

Theorem (R Bryant ~2000)**:**

The rolling dist for a pair of balls with $\rho = R/r \neq 1$ is *flat* (ie a CE dist)

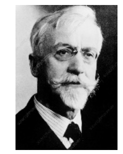
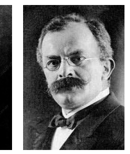
$\Leftrightarrow \ \rho = \frac{1}{3}$ or 3

(for $\rho \neq 1, 3, 1/3,$ sym gp is 6-dim)

"Rigid" curves of rolling distributions: rolling along **geodesics** (great circles)

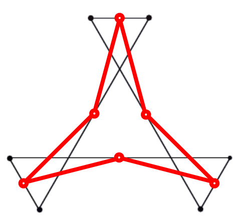
R Bryant, L Hsu (1993), *Rigidity of integral curves of rank 2 distributions*, Invent. math. 114

GB, R Montgomery (2009), *G_2 and the 'rolling distribution'*, Enseign. Math. 55

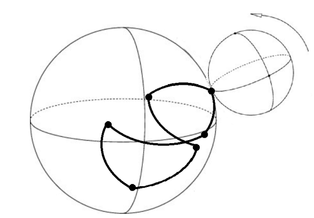
 

$$\widetilde{Q}^{\texttt{oct}}$$

2:1 ↙ ↘ 2:1



$$Q^{\texttt{dance}} \subset Q^{\texttt{oct}} \qquad\qquad Q^{\texttt{roll}}$$



$$S^2 \times S^3$$

$(\pm 1,\ 1)$ ↙ ↘ $(1,\ \pm 1)$

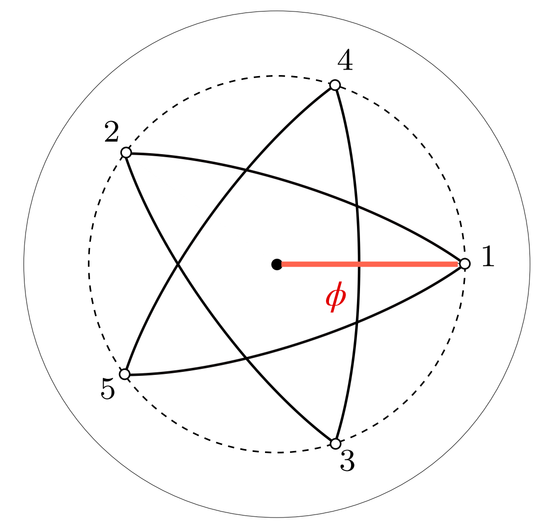
$$\mathbb{RP}^2 \times S^3 \qquad\qquad S^2 \times \mathrm{SO}_3$$

Spherical **regular** n-gons with trivial rolling monodromy

Proposition

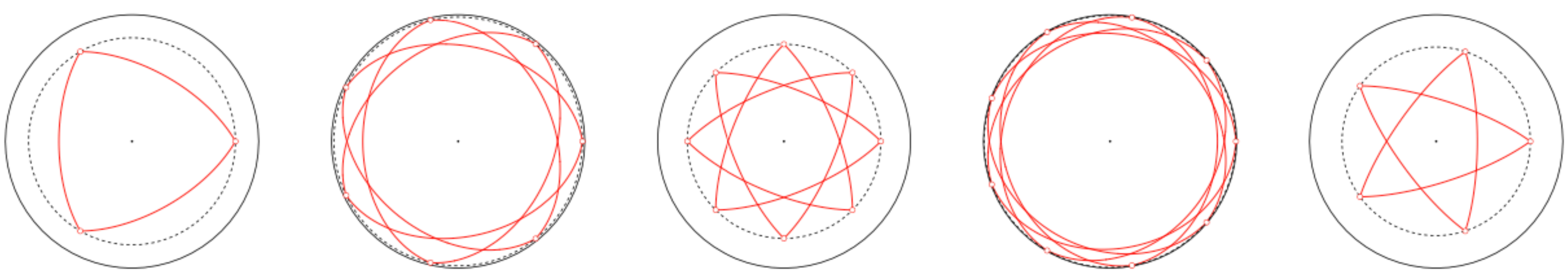
(a) A regular spherical polygon (n, w, ϕ) has trivial 3:1 rolling monodromy iff there exists an integer w' such that

- $$\cos\left(\frac{\pi w'}{n}\right) = \cos\left(\frac{\pi w}{n}\right)\left[1 - 4\sin^2\left(\frac{\pi w}{n}\right)\sin^2\phi\right].$$
- $w \equiv w' \pmod 2$



$n = 5,\ w = 2$

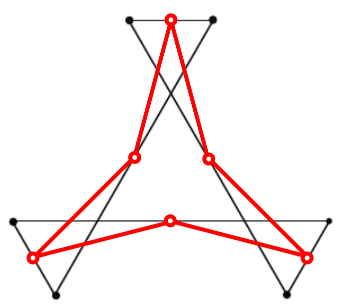
(b) There are solutions iff $n \geq 6$

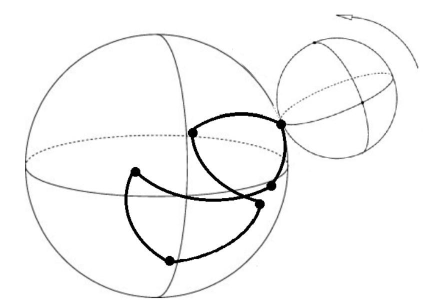


$(6, 2, 4)$ $(7, 2, 4)$ $(8, 3, 5)$ $(9, 2, 4)$ $(10, 4, 6)$

Theorem (main): there is a 1:1 correspondence

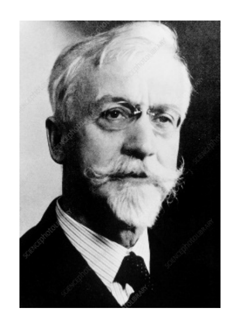
{ Dancing pairs of planar polygons } ⟷ { Spherical polygons with trivial rolling monodromy }

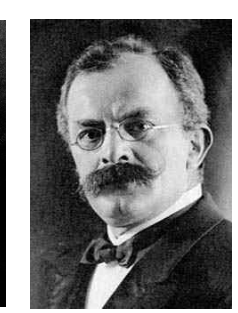




Proof: ↖↘ ↗↙

{ Closed piece-wise rigid Horizontal curves of the Cartan-Engel distribution }





Thank you!!